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Resonances in random binary optical media

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Abstract. We show that a succession of N random optical layers, whose optical index can assume only two values, allows for a perfect transmission (resonances) at fixed frequency values. The resonant frequencies are computed analytically for a generic probability distribution of disorder and for an arbitrary incidence angle. The arguments can be extended to probability distributions with more than two values.

Since the pioneering work of Anderson [1] and Mott [2], the localization properties of the solutions of the one-dimensional (1D) Schrödinger equation in the presence of random potentials have been widely studied. The results of many numerical and analytical works show that in a generic 1D random potential all electronic states are localized. This localization is a quantum mechanical phenomenon and takes place for arbitrary weak random potentials, if the hopping term is of a sufficiently short range. For a review of the localization problem see, e.g., [3].

Nevertheless, this does not exclude the possibility of random potentials that allow for extended states. Indeed Denbigh and Rivier [4], and more recently Crisanti *et al* [5], have shown not only that these potentials exist, but that they form an infinite class. However these are, in some sense, special 1D random potentials. Thus we can ask if, given a generic random potential, there can be some combinations of the parameters that lead to extended states *regardless* of the randomness.

In this paper we analyse this question in the context of the propagation of monochromatic light beams in 1D random optical media. Although Anderson localization occurs in quantum-mechanical problems, the phenomenon is essentially due to the wave nature of the electronic states, and thus could be found in any wave phenomena. There are indeed several experiments which show localization of photons [6] and also phonons [7] in random media. The optical experiments have the advantage that it is feasible to construct the system accurately and the parameters may be precisely controlled and measured [8]. The localization properties can then be easily extracted from the measurement of transmission through the medium. In particular an 'extended' state will result in a peak, also called a 'resonance', in the transmission.

Let us then consider a 1D random optical medium made by a sequence of layers of thickness a and optical index n_i. For each layer the value of n_i is extracted according a given probability distribution and 'quenched'. This can be regarded as a random Fabry–Perot interferometer [8]. A monochromatic light beam is then sent on the medium with an incidence angle θ_0. We remember that the incidence angle θ_0 is

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defined as the angle between the light beam and the normal to the surface of incidence. Previous theoretical study of this model considered continuous distributions of n_i, with [9, 10] or without [11] spatial correlations. By analogy with the localization problem, in this case the penetration length ξ of the light beam in the medium is always finite. In the case of an incident angle equal to the critical angle, the problem becomes formally equivalent to localization in a 1D discrete Schrödinger equation at the band edge $E = -2$ of the pure system [9, 11].

From an experimental point of view, continuous distribution are, however, hypothetical situations. In fact, in real experiments n_i can assume only a finite number of discrete values; usually two [12]. For this reason here we consider an uncorrelated binary distribution for the optical index. In other words, n_i can assume with equal probability only two values. However our analysis can be extended to probability distributions with any number L of values. In particular, continuous distributions can be formally obtained in the limit $L \to \infty$. Without losing generality, we may take $n_i = 1$ and $n_i = n$. This problem, for the case of a light beam perpendicular to the surface of the first layer, i.e. $\theta_0 = 0$, was studied by Flesia [10], who noted resonances for some values of the frequency of the incident beam.

Here we generalize the results to the general case of a light beam falling on the optical medium at an arbitrary incidence angle θ_0. We are not able to compute analytically the complete form of the penetration length as a function of the parameter. However, we can calculate analytically the frequency values of the resonances. We stress that these resonances exist *regardless* of the disorder.

To set up the problem, let us consider a monochromatic light beam, sent from an embedding medium with optical index n_0, falling on a 1D random medium made of $i = 1, 2, \ldots, N \gg 1$ layers of thickness a and optical index n_i. Without losing generality we can set $a = 1$. It is a straightforward exercise to show that the (complex) transmission and reflection coefficients t_N and r_N of the N layers are related by

$$\begin{pmatrix} t_N \, \mathrm{e}^{\mathrm{i}k_0 N} \\ \mathrm{i}k_0 \, t_N \, \mathrm{e}^{\mathrm{i}k_0 N} \end{pmatrix} = \left(\prod_{1=1}^{N+1} \mathbf{Q}_i \right) \begin{pmatrix} 1 + r_1 \\ \mathrm{i}k_0 (1 - r_1) \end{pmatrix} \tag{1}$$

where

$$\mathbf{Q}_i = \begin{pmatrix} \cos k_i & k_i^{-1} \sin k_i \\ -k_i \sin k_i & \cos k_i \end{pmatrix} \tag{2}$$

k_0 is the wavenumber of the light beam in the embedding medium, and

$$k_i^2 = (\omega/c)^2 \left(n_i^2 - n_0^2 \sin^2 \theta_0 \right) \tag{3}$$

is the wavenumber in the ith layer. Here ω is the light beam frequency, c the speed of light in vacuum and θ_0 the incidence angle (which is assumed to be in the interval $[0, \frac{1}{2}\pi)$). We assume that there is no absorption, so that $|t_N| + |r_N| = 1$. We are interested in studying how the penetration length ξ, defined as

$$\xi^{-1} = - \lim_{N \to \infty} \frac{1}{N} \ln |t_N| \tag{4}$$

changes as function of the frequency ω.

From the theory of products of random matrices we have that

$$|t_N| \sim \mathrm{e}^{-N\lambda} \qquad N \gg 1 \tag{5}$$

where λ is the maximal Lyapunov exponent of the product $\prod_i \mathbf{Q}_i$, so that $\xi_p = 1/\lambda$. Note that the Oseledec theorem [13] ensures that λ, and hence ξ, is a non-random quantity in the sense that, for almost all realizations of disorder, it does not depend on the particular realization. This is why we did not perform any average in (4).

Let us assume for the moment that n_i is constant, namely $n_i = \overline{n} < n_0$. It is well known that in this case there exists a critical incidence angle, given by $\sin\theta_c = \overline{n}/n_0$, such that the transmission coefficient, as a function of the penetration depth, changes from an oscillatory behaviour ($\xi^{-1} = \lambda = 0$) for $\theta_0 < \theta_c$, to an exponential decay ($\xi^{-1} = \lambda > 0$) for $\theta_0 > \theta_c$. In terms of random matrices this corresponds to having expanding matrices, i.e. with the largest eigenvalue of modulus greater than one, for $\theta_0 > \theta_c$ ($k^2 < 0$), and marginal matrices, i.e. with eigenvalues of modulus equal one, for $\theta_0 < \theta_c$ ($k^2 > 0$).

In our case there will be in general two critical angles given by

$$\sin\theta_a = 1/n_0 \qquad \sin\theta_b = n/n_0.$$

Obviously the existence of these angles depends on n and n_0. In what follows we implicitly assume that $1 < n < n_0$ so that both critical angles exist and $\theta_a < \theta_b$. Depending on the incident angle θ_0 the wavevector k can be real or pure imaginary [see (3)], corresponding to marginal and expanding matrices $\mathbf{Q}_i$, respectively. We have, therefore, three possible cases: all matrices are marginal, some are marginal and some are expanding, and all are expanding matrices. Similar situations are found in 1D localization, where the role of θ_0 is played by the energy of the electrons.

In our case there are only two possible matrices $\mathbf{Q}_i$ corresponding to $n_i = 1, n$. Thus it is useful to denote them by $\mathbf{A}$ ($n_i = 1$) and $\mathbf{B}$ ($n_i = n$). Since there is no need of distinction, in what follows $\mathbf{A}$ ($\mathbf{B}$) will indicate both the layers with $n_i = 1$ ($n_i = n$) and the matrices associated with them. Similarly we indicate by k_a (k_b) the wavenumber in the layers $\mathbf{A}$ ($\mathbf{B}$).

To find where one could have resonances let us consider the physics of the problem. When the light beam reaches the boundary between two successive layers, a fraction t will penetrate inside the next layer while a fraction r ($|r| = 1 - |t|$) will be reflected back. This will happen regardless of the incident angle at the specific boundary. The only difference is that when the angle becomes larger than the critical incidence angle of the boundary, the wave will be exponentially damped. This in terms of matrices corresponds to having an expanding $\mathbf{Q}$. Therefore, if $\theta_0 > \theta_b$ the penetration length ξ is always finite since the light is always damped. This is easily seen from (4), since now all matrices are expanding. In the range $\theta_a < \theta_0 < \theta_b$ there will be a finite fraction of expanding and marginal matrices arranged in a random order. Thus in general ξ will be again finite. Let us stress, however, that this is the range where one can apply the theory of [5] to build random sequences of layers which allow for infinite penetration length. Here, one of the matrices $\mathbf{A}$ and $\mathbf{B}$ is marginal and one is expanding, thus if they were arranged in a periodic order, one could apply a sort of Bloch theorem obtaining bands where $\xi = \infty$. The penetration length, therefore, can be made infinite at selected frequencies by using the method of [5] to build the random sequence of layers.

Note that even if ξ is always finite in a generic random case, we expect that its behaviour would resemble 'qualitatively' that of a periodic ordering. In our case, since **A** and **B** are uncorrelated, the relevant periodic order is **ABABAB** $\cdots$. Indeed a numerical calculation leads to a ξ-behaviour for the random case very similar to that observed in the above periodic order, see figure 1. Note that, even if ξ is always finite, there are peaks for ω values inside the bands of the ordered system.

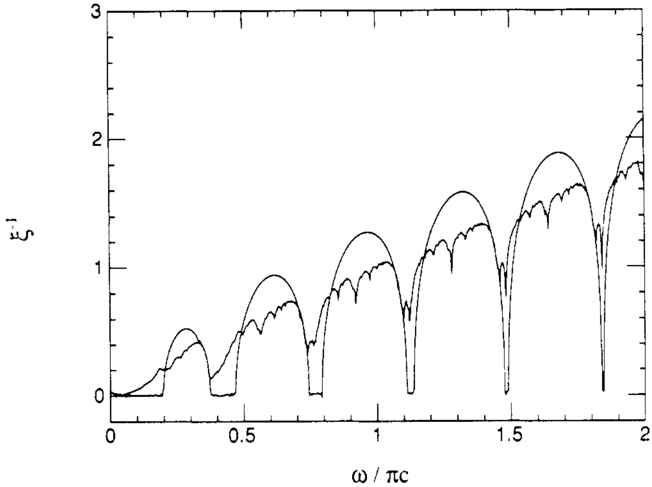

Figure 1. ξ^{-1} as a function of ω for $\theta_a < \theta_0 < \theta_b$. The results are compared with behaviour for the periodic ordering **ABAB** $\cdots$ (smooth curve). The parameters used are: $n_0 = 2$, $n = 3$ and $\theta_0 = 0.6$.

From the above discussion it follows that the only range where one could find resonances is $0 \le \theta_0 < \theta_a$, where the light beam penetrates in all layers without damping. This, however, is not a sufficient condition for an infinite ξ since the components reflected back at each boundary layer interfere with the propagating beam. The result of the interference depends on the difference of optical paths between the two beams, which results in a phase difference. Since the order of the layers is completely random, the interference is in general destructive, leading to a complete decay of the light wave over a finite distance ξ. This is a well known result in the theory of random matrices and 1D localization.

However, since at fixed optical indexes, the length of the optical path depends continuously on the light frequency, there may be ω values for which the interference becomes constructive. We then say that there is a resonance, and $\xi = \infty$. A simple inspection of the matrix $\mathbf{Q}_i$, shows that a trivial resonance is obtained for $k_i = 0$, i.e. $\omega = 0$. This, however is not a very interesting case. The question is, therefore, are there non-trivial ω values for $\theta_0 < \theta_a$ which satisfy a condition for a constructive interference? The physics of the problem suggests that if there are resonances, they can appear only when ω is such that one of the two types of layers (e.g. **A**) contains exactly an integer or a semi-integer number of wavelengths of the light. In fact in this case the effect of these layers is just an overall phase $\exp(i\alpha)$ and can be disregarded. In terms of random matrices this means that in the product (4) we are left only with the matrices **B**, so that $\xi^{-1} = \rho\lambda_B = 0$, since the matrix **B** is marginal. Here ρ is the density of layers **B** in the N layers medium and λ_B is the maximal Lyapunov exponent of the product of **B** matrices. Thus there will be a resonance every time the monochromatic light wave $\psi_k(x)$ satisfies $\psi_k(x+1) = (-1)^p \psi_k(x)$ in one of the two types of layers. This leads to the resonance conditions k_a or k_b equal to $p\pi$ with $p = 1, 2, 3, \ldots$.

A second type of resonance can be present. These could appear when the wavefunction changes by a factor ± 1 over two consecutive layers. Since the order is completely random, we need to consider only the cases **AA**, **BB** and **AB**. The analysis of this case leads to resonances for $k_a = p\pi/2$ and $k_b = q\pi/2$ with p, q odd integers. The demonstration in this case is less direct and will be not reported here.

Finally there are special resonances when θ_0 is equal to the lower critical angle θ_a. We call these 'critical resonances', and are obtained for $k_a = 0$ and $k_b = p\pi$ with $p = 1, 2, 3, \ldots$. Note that there are no resonances if $k_b = p\pi/2$ with p an odd integer. Indeed in this case it can be demonstrated with the help of the theory of products of random matrices that ξ is finite. In particular, if $p\pi/2 \gg 1$ then

$$\xi^{-1} = \frac{1}{2\pi^2 p^2} - \frac{9}{4\pi^4 p^4} + \mathrm{O}(p^{-6}) \tag{6}$$

which is in a very good agreement with numerical calculations even for small values of p. For example, for $p = 1$ the above formula gives 0.027 56 and the numerical value is 0.042 05, whereas $p = 3$ (6) gives 0.005 62 and from numerical simulations we get 0.005 34. This result has been obtained with the microcanonical method recently developed by Deutsch and Paladin [14]. As far as we know this is the first example of an analytical result obtained with this method. The calculation will be reported elsewhere.

Since the probability of having **A** or **B** are completely uncorrelated, these are all the possible resonances. It is easy to express the above result as conditions on ω. We then have for $0 \le \theta_0 < \theta_a$

$$\frac{\omega}{\pi c} = \begin{cases} p\chi_a & p = 1, 2, 3, \ldots & (7a) \\ p\chi_b & p = 1, 2, 3, \ldots & (7b) \\ p\chi_a/2 = q\chi_b/2 & p = 1, 3, \ldots \quad q = 1, 3, \ldots \ge p & (7c) \end{cases}$$

where $\chi_a^{-1} = (1 - n_0^2 \sin^2\theta_0)^{1/2}$ and $\chi_b^{-1} = (n^2 - n_0^2 \sin^2\theta_0)^{1/2}$. At the critical angle θ_a the resonances (7*a*) disappear. These results are in very good agreement with numerical calculations. In figure 2 we shown a typical ξ-behaviour in the range $0 \le \theta_0 \le \theta_a$.

From the above analysis it is clear that these resonances exist *regardless* of the disorder. Thus they will be in general present for any binary distribution of optical indexes, with or without spatial correlations. Nevertheless, we note that for some special distributions other resonances could appear. These, however, are not generic, in the sense that they depend on the particular probability distribution.

To conclude, we stress that our results can be extended to distributions of n_i with more than two values. In this case, however, the number of constraints to be satisfied to have a constructive interference is higher. Thus in general we expect a decrease in the number of resonances. This is in agreement with recent experimental results on conduction in disordered superlattices [15]. In the limit of a continuous distribution the number of constraints to be satisfied become infinite and no resonances are found, in agreement with the results of [9, 11, 15]. Here we mean resonances in the strict sense, i.e. $\xi^{-1} = 0$. In fact, if one allows for a small continuous spread about the two values of n_i, ξ is finite. Nevertheless it exhibits peaks at ω values given by (7), whose values diverge as the spread width goes to zero. Therefore if the medium is too short one could still observe transmission of light.

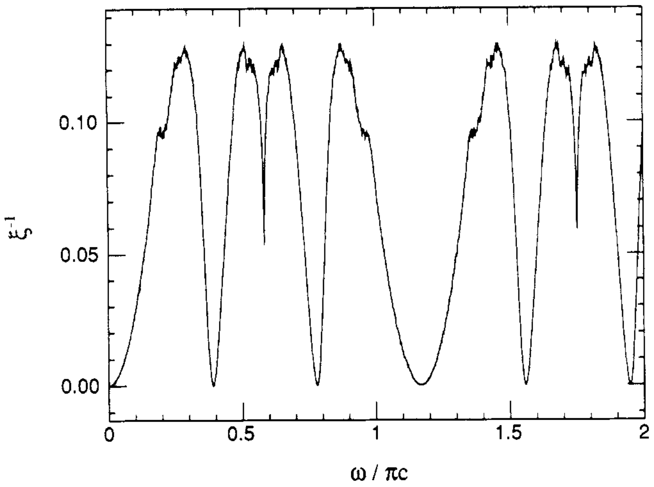

Figure 2. ξ^{-1} as a function of ω for $\theta_0 < \theta_a$. The parameters used are: $n_0 = 2$, $n = 8\sqrt{3} - 7 \simeq 2.618$ and $\theta_0 = \pi/12$. One can recognize the resonances (7*a*) with $p = 1$; (7*b*) with $p = 1, 2, 3, 4$; (7*c*) with $p = 1, 3$ and $q = 3p$. Note that the resonance (7*a*) is degenerate with the resonance (7*b*) with $p = 3$. In this case, in fact, all the resonances (7*a*) are degenerate with the resonance (7*b*) with $p = 3n$ where n is a positive integer.

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